**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

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| *What is this*?  |

**The aim of this course** is solving of different extremum problems.

The course consists of two parts:

* ***Calculus of variation*** that is the methods of solving of minimization problems for integral functionals.
* ***Optimization methods***that is the methods of solving of optimization control problems.

# Introduction

The introduction of the course consists of two lectures:

* **Practical examples of the extremum problems**. We consider three typical practical examples of the extremum theory problems.
* **Minimization of functions.**We get the stationary condition. This is the easiest standard method of finding the extremum of the functions and the basis of extremum theory.

## Lecture 1. Practical examples of the extremum problems

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| *Why it can be useful*?  |

We have the extremum theory. Consider at first examples of practical extremum problems.

* **Length** **maximization of the body flight.** This is a problem of the maximization for a function.
* **Brachistochrone problem.** This is a problem of the calculus of variations. We have the minimization problem for an integral functional.
* **Length** **maximization of the missile flight.** This is an optimization control problem.

The problems of the minimization or maximization for the functions are the basis of the extremum theory, and the variations calculus problems and the optimization control problems are the general classes of this theory.

### 1.1. Length maximization of the body flight

We throw a body with the unknown angle of the throw. We would like to find this angle such that the length of the body flight will be maximal. At first, we give the problem statement.

By the *second Newton’s low* we have the following equations

  (1.1)

where *t* is the time that is the *independent variable*, *x* and *y* are the horizontal and the vertical coordinates of the body that is the state that are the *state functions*, *m* is its mass, and *P* is its weight that are the given *parameters of the system* (see Figure 1.1). We consider these differential equations with the initial conditions

  (1.2)

where the velocity *v* is given, and the angle of the throw *ϕ* is unknown. The *Cauchy problem* (1.1), (1.2) is the *mathematical model* of the system.

We have also the final condition of the flight

  (1.3)

where *T* is a final time. It is obvious that the length of the flight depends on the angle. Calculate it by the formula (see Figure 1.1).

  (1.4)

Thus, we have the following problem.

**Problem 1.1**. *Find the angle ϕ such that the length L will be maximal.*



Figure 1.1. The flight of the body.

Transform this problem statement. Determine the solution of the Cauchy problem (1.1), (1.2). Integrate the equations (1.1) with known initial state derivatives. We have

  (1.5)

where *g* is the gravitational acceleration. Using (1.2), (1.5), we find the state functions



Using (1.3), we obtain the equation with respect to the final time



Then we get



Using (1.4), we find the length of the flight

 . (1.6)

Thus, Problem 1.1 is transformed to the problem of maximization of the function *L* with respect to the variable *ϕ* . We obtain the *maximization problem for a function*. This is the easiest extremum problem. We shell solve it in the next lecture.

### 1.2. Brachistochrone problem

We have two points *A* and *B* of the vertical plane. We would like to find the curve
*у* = *у*(*х*) with the begin *A* and the end *B* such that the time of the movement by the influence of the weight only is minimal (see Figure 1.2). Give the mathematical problem statement.



Figure 1.2. The movement of the body.

Let the point *A* be the origin of coordinates; and the point *B* has the coordinates (*Х*,*У*). Therefore, our curve satisfies the boundary conditions

 *у*(0) = 0, *у*(*X*) = *Y*. (1.7)

By the second Newton’s low we have

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Find the velocity of the body at the point with coordinate *y*. We have

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Then we get



Hence, we obtain the equality

*v* *dv* = *g* *dy*.

Find the general solution of this differential equation. After integration we have

*v*2/2 = *g y* + *c*,

where *с* is an arbitrary constant. The velocity at the point *A* with the coordinate *у* = 0 is zero. Then we obtain



Let *s* be the way of the body from the point *A* to the point *M* by the curve (see Figure 1.2). Then the velocity is the derivative

  (1.8)

Consider the point *M* with the coordinates (*х*,*у*) at the time *t*, and the point *M*' with the coordinates (*х* + Δ*x*, *у* + Δ*y*) at the time *t*+Δ*t* (see Figure 1.3).



Figure 1.3. The way of the body.

Let the interval Δ*t* be small enough. Then we have



After passing to the limit as Δ*t* →0 we obtain the equality



where *y*' = *dy*/*dx*.

Using (1.2), we get

  (1.9)

The body locates in the point *A* with coordinate *х* = 0 at the time *t* = 0; and it is in the point *B* with coordinate *X* at the time *T.* After integration we have



Hence the time of the movement from the point *A* to the point *B* is



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| **Question**: *What kind of the dependence we have?* |

The argument *y* is a function. For all *y* the value  is a number. Therefore, the mapping  is the functional. Thus, we have the following extremum problem.

**Problem 2.** *Minimize the functional T with condition* (1.7).

This is the minimization problem of the integral functional. This is the problemof *the* *calculus of* *variations.* We shell solve the similar problems in the first part of this course (see Lecture 4).

### 1.3. Length maximization of the missile flight

We consider now the flight of the missile in the vertical plane (see Figure 1.4). The missile moves by the driving force during the given time interval . Then it moves by the gravitation only because the fuel finished. Using the Newton’s low, we have the equations



where *m* is the mass, *Р* is the weight, *Fx* and *Fy* are the horizontal and vertical driving forces (see Figure 1.4). We determine the forces

*Fx* = *F* cos *u*, *Fy* = *F* sin *u*, *P* = *mg*.

Then we get

  (1.10)

where the acceleration *а = F/m* is known. The function *u* is the *control* here because we can choose it.



Figure 1.4. The flight of the missile.

We have the initial conditions

  (1.11)

If the function *u* **=** *u*(*t*) is given, then we can find the solution of the Cauchy problem (1.10), (1.11) and find the characteristics of the movement at the final time *T* that is the end of the active flight, because the fuel ends.

  (1.12)

For the time *t* >*T* we have the movement by the gravitation only. Therefore, we have the equations



Find its general solution

*x*(*t*) = *α*1 *t* + *β*1, *y*(*t*) = –*g t2*/2 + *α*2 *t* + *β*2,

where unknown constants *α*1, *β*1, *α*2, *β*2 can be found with using of the conditions (1.12). Then we obtain

*α*1 = *uT*, *β*1 = *xT* – *uT T*, *α*2 = *vT* + *gT*, *β*2 = *yT* – *vTT* – *gT2* /2.

Therefore, we have the low of the movement

 *x*(*t*) = *xT* + *uT* (*t* – *T*), (1.13)

 *y*(*t*) = *yT*+ *vT* (*t* – *T*) – *g* (*t* – *T*)2 / 2. (1.14)

The vertical coordinate equals to the zero at the final time *t* = *τ*. By (1.14), we have the quadratic equation

*g*(*τ* – *T*)2 / 2 – *vT* (*τ* –*T*) – *yT*= 0.

Then we find its solutions



The difference *τ* –*T* is positive; therefore, we chose the sign “+” here.

Using (1.13), we find the length of the flight

*L* = *x*(*τ*) = *xT* + *uT* (*τ* – *T*) = *xT* + *uT* ** /** *g* .

Now we can give the problem statement.

**Problem 3**. *Chose the function* *u* **=** *u*(*t*) *for minimization the length of the flight*

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*where the functions x and y are the solutions of the problem* (1.10), (1.11).

Now the maximizing functional depends from the unknown function *u* non-directly. The functional depends from state functions. There are the coordinates and the velocities of the missile that depend from *u* by the state equations. This is the *optimization control problem*. We shell solve similar problems in the second part of this course (see Lecture 11).

### Outcome

* Many natural phenomenon’s can be transformed to extremum problems.
* The aim of this course is the analysis of the problems of minimization or maximization for the functions, variational problems and optimization control problems.
* Our first step is the general method of finding of the extremum for the functions.
* It will be the basis for the calculus of variations and the optimization control theory.

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### Next step

The aim of our course is solving of extremum problems. The easiest extremum problem is the problem of minimization of the function of one variable. Therefore, we would like to have the general method of functions minimization. It will be the basis of the extremum theory.